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AUTOMATIC PROCESS CONTROL IN A GENERAL MARKOVIAN SET UP

by

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Abstract

In automatic process control, it is sometimes realistic to assume that the assignable variation affecting the operating level of the process at every epoch depends on the current value of the process. In such situations, the conditions for system stability are investigated, when a 'two-sided controller' is used. An algorithm is provided here to obtain the steady state distribution of the observations of the process. Three numerical illustrations are also presented.

Keywords: process control, two-sided controller, Markov chains.

1. Introduction

We have shown in an earlier work (Sarkar, and Bhattacharji, 1986) that a two-sided controller with a properly chosen parameter stabilises a high speed production process subject to a specific type of assignable variation. It is assumed there that the amount of shift from the current value of the process operating level (Process mean) produced by assignable causes follow a Normal distribution with mean zero and a known variance, irrespective of the current value of the process.

In the present paper, we consider a more general type of assignable variation which follows a probability law dependent on some suitable function of the current value of the process mean. The conditions under which a 'two-sided controller' induces stability into the process in such a situation are investigated here. A method, along with an algorithm, for evaluating the steady state distribution of the observations is outlined. The analysis provides an insight into the behavior of the process so that the implementation of such a control device can be carried out easily.

2. The process model and the probability laws of the random variables.

The control device and the process model considered here are formally identical to those of the earlier work (Sarkar and Bhattacharji, 1986). Thus we suppose that assignable causes shift the level of the process mean y_n at every epoch n, by a random amount a_n, independently of time. The control device gets activated and shifts y_n, by a random amount s_n only when the observation x_n at epoch n goes beyond given lower and upper control limits C_* and C^* respectively. So long as observations on the process remain within the two control limits no action is taken. Denoting the random variables corresponding to x_n, y_n, a_n and s_n by the corresponding capital letters the process model, therefore, takes the following form

$$X_n = Y_n + \epsilon_n$$

$$Y_n = Y_{n-1} + A_{n-1} + S_{n-1} \quad , \quad n \geq 1 \qquad (2.1)$$

with initial value $Y_o = 0$, and $E[\epsilon n] = 0$. It is assumed without much loss of generality that the random variables Y_n, An and Sn take discrete values 0, ±1, ±2, ...

It is assumed that the probability law on X_n, when the process mean has a value y_n is $N(y_n, \sigma^2)$, and that of Sn for every epoch n, is

$$\Psi_s(s|y_n) = G_s(S|C_*)\ F_{C*}(y_n) + G_s(S|C^*)\ (1-F_{C*}(y_n))$$
$$+ G_s(s|C^*-C_*))\ (F_C{}^*(y_n) - F_{C*}\ (y_n)) \qquad (2.2)$$

where $G_s(s|C_*)$, $G_s(s|C^*)$ and $G_s(s|(C^*\_C_*))$ are the conditional probability mass function of S, given that

$$x \leq C_*, \quad x \geq C^* \quad \text{and} \quad C_* < x < C^* \quad \text{respectively,}$$

and $$F_C(y_n) = \int_{-\infty}^{C} \frac{1}{\sqrt{2\pi\sigma}} \exp\{ \frac{-1}{2}(\frac{x-y_n}{\sigma})^2\}\ dx.$$

Furthermore, it is assumed here that the probability law of An when the process mean has a value y_n is a function of y_n. For example, if the assignable variation follows a normal law, it is given by

$$P(A_n = a) = \varphi_A\ (a = j_\alpha / Y_n)$$

$$= \int_{(j-.5)^\alpha}^{(j+.5)\alpha} \frac{1}{\sqrt{2\pi}\sigma_\alpha} \exp\left(-\left(\frac{t-z(Y_n)}{\sigma_\alpha}\right)^2 \right) dt \qquad (2.3)$$

where α is a small positive constant, $j = 0, \pm 1, \pm 2, \ldots.,$ and $\mu = z(y_n)$ is some function of yn.

Thus when the process mean is at 2, with $\mu = 2$ and $\sigma_\alpha = 1$, we obtain for $j = 0, \pm 1, \pm 2, \pm 3, \ldots.$

$$\varphi_\alpha(j^\alpha | 2) = \Phi((j+.5)\alpha - 2) - \Phi((j-.5)\alpha - 2)$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal variable.

Since A_n and S_n take discrete values, from (2.1), (2.2) and (2.3) it follows that the unobservable stochastic process reduces to a Markov chain on state space

$$C_2 = [\ldots., -2, -1, 0, 1, 2, \ldots.] = [m | m \in I\ ]$$

where I is the set of integers.

The one-step stationary probability $p_{m,m+r}$ of transition from state m to state m+r is given by

$$p_{m,m+r} = P(\ Y_{n+1} = m+r \mid Y_n = m) = \quad _r(m), \text{ say,}$$

$$r = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

The $\xi_r(m)$'s are given by the convolution of $\{\varphi_r(m)\}$ and $\{\ _{r'}(m)\}$, that is

$$\{\xi r(m)\} = \{\varphi_r(m) * \{\xi_r(m)\} \qquad (2.4)$$

Now that $_r(m)$'s become explicit function of m whatever be the value of m and it is in this respect that the resent work differs from the earlier one. It is physically quite meaningful to consider the present form of $\varphi_r(m)$. The implication of (2.3) is that the characteristics of the assignable variation may be affected to a greater or lesser extent by the current value of the process mean. Such an assumption may be reasonable in many practical situations. For example, in some types of production processes (chemical, metallurgical, etc.) temperature control, which is one of the major components of the operating level of the process, may get affected by the current level of some other component like the amount of impurities in the inputs (reagents/ores). When the operating level of the latter has a tendency to rise it may cause a rise in the temperature also which in effect may bring a disturbance in the functioning of the process. These disturbances affecting the operating level of the process, may then be supposed to follow a probability law depending on the current value of the process.

Thus the present model may be regarded as a substantive generalization of the earlier one. The analysis in this case will also be significantly different.

In the next section, we investigate conditions on $\xi_r(m)$'s for Markov chains on $C_1 = \{0, 1, 2, \ldots\}$ and C_2 to be positive recurrent. These conditions are the conditions for stability of the process. For some sufficient conditions on ergodicity of irreducible aperiodic Markov chains on countable or more general state spaces reference may be made to Pakes(1969) and Tweedie(1975) among others. The sufficient conditions that we have derived here is quite convenient for the purposes of the present study.

3. Ergodic behavior of Markov chains on state spaces C_1 and C_2

We consider a Markov chain on state space C_1 with transition probability matrix P where (i,j)th element is $_{j-1}(i)$. Without loss of generality the chain may be supposed to be aperiodic and irreducible. The Markov chain is ergodic if and only if the system of linear equations

$$u = P^T u \tag{3.1}$$

where u is the column vector $[u_0, u_1, u_2, \ldots]^T$ and P^T is the transpose of P, has a solution u_m satisfying

$$u_m \geq 0$$

and

$$\Sigma\, u_m < \infty \tag{3.2}$$

(cf. Feller, 1967).

Since $\xi_r(m)$'s for each r is bounded from above and below, it has a lim sup and a lim inf say Δ_r and δ_r respectively. We assume that for at least one value of r, $\delta_r, \neq \Delta_r$ and $\delta_r > 0$ for at least one positive r and one negative r. So for all sufficiently large m,

$$0 \leq \delta_r - \epsilon \leq {}_r(m) \leq {}_r + \epsilon \leq 1$$

where ϵ is arbitrarily small. We also assume that given any $\epsilon > 0$, we can find integers r_1, r_2 such that

$$\sum_{r=r_1}^{r_2} r(m) \geq 1 - \epsilon .$$

So $\sum_{r=-\infty}^{\infty} {}_r < \infty$, $\sum_{r=-\infty}^{-r_1-1} {}_r < \epsilon$ and $\sum_{r=r_2+1}^{\infty} {}_r < \epsilon$.

We also have, $0 < \sum_{-\infty}^{\infty} \delta_r < 1.$

In order to determine whether the system of equations (3.1) with variable coefficients has a solution satisfying the condition (3.2), we consider the system of difference equations with constant coefficients

$$\sum_{\substack{r=-r_1 \\ r\neq 0}}^{r_2} \Delta_r u_m = \sum_{\substack{r=-r_1 \\ r\neq 0}}^{r_2} \delta_r u_{m-r} \tag{3.3}$$

and

$$\left(\sum_1^{r_1} \delta_{-r} + \sum_1^{r_2} \Delta_r \right) u_m = \sum_1^{r_1} \delta_{-r} u_{m+r} + \sum_1^{r_2} \Delta_r u_{m-r} \tag{3.4}$$

The characteristic equations of (3.3) and (3.4) are respectively

$$X(Z) = \sum_{r=1}^{r_1} \delta_{-r} Z^{r_2+r} - \left(\sum_{r=r_1}^{r_2} \Delta_r \right) Z^{r_2} + \sum_{r=1}^{r_2} \delta_r Z^{r_2-r} = 0 \tag{3.5}$$

and

$$\beta(Z) = \sum_{r=1}^{r_1} \delta_{-r} Z^{r_2+r} - \left(\sum_{r=1}^{r_1} \delta_{-r} + \sum_{r=1}^{r_2} \Delta_r \right) Z^{r_2} + \sum_{r=1}^{r_2} \Delta_r Z^{r_2-r} = 0 \tag{3.6}$$

Each has two changes of sign, and so has at most two positive roots.

Since $X(0) > 0$ and $X(1) < 0$, $X(Z)$ has exactly two positive roots, say, $0 < s_1 < 1$ and $s_2 > 1$.

Since $\beta(0) > 0$ and $\beta(1) = 0$, $\beta(Z)$ has unity as a root and hence has exactly one more positive root, say p.

From (3.5) and (3.6) we have

$$X(0) > 0 \quad X(s_1) = 0 \quad X(p) < 0 \quad X(1) < 0 \quad X(s_2) = 0$$

$$\beta(0) > 0 \quad \beta(s_1) > 0 \quad \beta(p) = \beta(1) = 0 \quad \text{and} \quad \beta(s_2) > 0$$

irrespective of whether p is less than, equal to or greater than unity

Suppose $p < 1$. Consider the iteration formula

$$y_m^{(n+1)} = \min \left[\frac{\sum_{r=1}^{r_1} \delta_{-r} \, y_{m+r}^{(n)} + \sum_{r=1}^{r_2} \xi_r (m-r) y_{m-r}^{(n)}}{\sum_{r=1}^{r_1} \xi_{-r}(m) + \sum_{r=1}^{r_2} \Delta_r}, \; p^m \right], \; n=1,2,..$$

with $y_m^{(0)} = p^m$, $y_0^{(n)} = 1$ for $n = 0, 1, 2, \ldots$ (3.7)

Then

$$y_m^{(1)} = \frac{\sum_{r=1}^{r_1} \delta_{-r} \, p^{m-r} + \sum_{r=1}^{r_2} \xi_r \, (m-r) \, p^{m-r}}{\sum_{r=1}^{r_1} \xi_{-r}(m) + \sum_{r=1}^{r_2} \Delta_r} \leq y_m^{(0)} = p^m$$

for sufficiently large m, say $m \geq M_0$

since

$$p^m = \frac{\sum_{r=1}^{r_1} \delta_{-r}\, p^{m-r} + \sum_{r=1}^{r_2} \Delta_r\, p^{m-r}}{\sum_{r=1}^{r_1} \delta_{-r} + \sum_{r=1}^{r_2} \Delta_r}$$, and $\epsilon > 0$ is arbitrary and

$$y_m^{(2)} = \frac{\sum_{r=1}^{r_1} \delta_{-r}\, y_{m+r}^{(1)} + \sum_{r=1}^{r_2} \xi_r\,(m-r)\, y_{m-r}^{(1)}}{\sum_{r=1}^{r_1} \xi_{-r}(m) + \sum_{r=1}^{r_2} \Delta_r} \leq y_m^{(1)} .$$

In general, for sufficiently large m

$$y_m^{(n+1)} \leq y_m^{(n)}, \qquad n = 0, 1, 2, \ldots .$$

Also

$$y_m^{(1)} = \frac{\sum_{r=1}^{r_1} \delta_{-r}\, y_{m+r}^{(0)} + \sum_{r=1}^{r_2} \xi_r\,(m-r)\, y_{m-r}^{(0)}}{\sum_{r=1}^{r_1} \xi_{-r}(m) + \sum_{r=1}^{r_2} \Delta_r}$$

$$\geq \frac{\sum_{r=1}^{r1} \delta_{-r}\, p^{m+r} + \sum_{r=1}^{r_2} \delta_r\, p^{m-r}}{\sum_{r=1}^{r_1} r + \sum_{r=1}^{r_2} r}$$ for sufficiently large m

$$\geq \frac{\sum_{r=1}^{r_1} \delta_{-r}\, s_i^{m+r} + \sum_{r=1}^{r_2} \delta_r\, s_1^{m-r}}{\sum_{r\neq 0} \Delta_r} = s_1^m$$ and

$$y_m^{(2)} = \frac{\sum_{r=1}^{r_1} \delta_{-r}\, y_{m+r}^{(1)} + \sum_{r=1}^{r_2} \xi_r\,(m-r)\, y_{m-r}^{(1)}}{\sum_{r=1}^{r_1} \xi_{-r}(m) + \sum_{r=1}^{r_2} \Delta_r}$$

$$\geq \frac{\sum_{\substack{r=-r_1 \\ r\neq 0}}^{r_2} \delta_r \; s_1^{m-r}}{\sum_{\substack{r=-r_1 \\ r\neq 0}}^{r_2} \Delta_r} = s_1^m \; .$$

In general for all sufficiently large m, $y_m^{(n)} \geq s_1^m$.

Thus the sequence $\{y_m^{(n)}\}$ is monotonically non-increasing for each m with a positive lower bound (which is s_i^m for sufficiently large m) and an upper bound p^m. So it converges to a positive limit, say y_m, which is at most equal to p^m, for each m.

Also, for all sufficiently large m,

$$s_1^m < y_m = \frac{\sum_{r=1}^{r_1} \delta_{-r}\, y_{m+r} + \sum_{r=1}^{r_2} \xi_r(m-r)\, y_{m-r}}{\sum_{r=1}^{r_1} \xi_{-r}(m) + \sum_{r=1}^{r_2} \Delta_r} < p_m \text{ for } m \geq m_0 \; .$$

Now consider the iteration formula

$$u_m^{(n+1)} = \sum_{r=-r_1}^{r_2} \xi_r(m-r)\, u_{m-r}^{(n)}$$

$$\text{with } u_m^{(0)} = y_m, \quad m=0,1,2,\ldots \tag{3.8}$$

Then for sufficiently large values of m, $u_m^{(1)} \geq u_m^{(0)}$.

So in general $u_m^{(n+1)} \geq u_m^{(n)}$.

Thus the sequence $\{u_m^{(n)}\}$, for sufficiently large values of m is non-decreasing and hence will tend to a positive limit u_m. This will ensure that $u_m^{(n)}$ tends to a positive limit for every value of m.

Let $u^{(n)}$ denote the column vector $[u_0^{(n)}, u_1^{(n)}, u_2^{(n)}, \ldots]^T$.

Then the iteration formula (3.8) can be presented as

$$u^{(n+1)} = P^T u^{(n)} = (P^T)^n [\, y_0, y_1, y_2, \ldots]^T .$$

where P is the transition probability matrix of C_1.

Since $\lim_{n\to\infty} u^{(n)} = u = [\, u_0, u_1, u_2, \ldots]^T > 0$, $\lim_{n\to\infty} (P^T)^n$

exists and is non-null.

Thus a sufficient condition for the ergodicity of C_1 is that equation (3.6) will have a positive root p, $0 < p < 1$. It was shown that (cf. Equation (5.5), Sarkar and Bhattacharji, 1986) $p<1$ if

$$\sum_{r=1}^{r_1} r\,\delta_{-r} > \sum_{r=1}^{r_2} r\,\Delta_r . \tag{3.9}$$

Thus (3.9) is a sufficient condition for ergodicity of the chain C_1. In case the chain C_1 is periodic we can use the iteration formula

$$u_m^{(n+1)} = \frac{u_m^{(0)} + u_m^{(1)} + \ldots + u_m^{(1)}}{(n+1)}$$

with $u_m^{(0)} = y_m$ in place of (3.8)

This will yield a solution to the system of equation.

For a Markov chain on C_2, it can be shown analogously, that the chain is positive recurrent if

$$\sum_{r=1}^{r_1} r\,\delta_{-r} > \sum_{r=1}^{r_2} r\,\Delta_r$$

$$\sum_{r=1}^{r_1} r\,\delta_r' > \sum_{r=1}^{r_2} r\,\Delta_{-r}', \qquad (3.10)$$

where δ'_r and Δ'_r are respectively the infimum and supremum of $\xi_r(m)$ as $m \to \infty$.

4. An algorithm for evaluation of steady state distribution of the observations.

I Choose a model of the type (2.3) with probability masses concentrated at j^α, $j = 0, \pm1, \pm2, \ldots$, α being a small positive constant.

II Choose control limits C_*/α, C^*/α and $G_s(s^\alpha|.)$.

III Obtain the $\xi_r(m)$'s by (2.4).

IV Check conditions of (3.10). If (3.10) is satisfied go to step V.

If (3.10) is not satisfied, the system does not attain stability.

V Use iteration formula (3.7) with $y_m^{(0)} = p^m$ for $m \geq 0$ and the formula

$$y_m^{(n+1)} = \min \left[\frac{\Sigma\, \xi_r(m-r)\, y_{m-r}^{(n)} + \Sigma\, \delta_r'\, y_{m-r}^{(n)}}{\Sigma\, \Delta_{-r} + \Sigma\, \xi_r(m)}, \eta^m \right]$$

with $y_m^{(0)} = m$ for $m < 0$, where η is the positive root (>1) of the equation corresponding to (3.6).

VI Use iteration formula (3.8) with $u_m^{(0)} = y_m$, $m=0, \pm1, \pm2, \ldots$ y_m's are the stabilised values of step V

VII Evaluate π_m by the equation $\pi_m = \dfrac{u_m}{\sum_m u_m}$, where u_m's are the stabilised values at step VI.

VIII Obtain the steady state distribution of the observations by

$$F(x) = \sum_m \pi_m \int_{-\infty}^{x} f_X(x/m)\, dx.$$

5. Ilustration

For illustrative purposes we consider three different cases. In the first two cases we take

$$\mu = \begin{cases} m^{1/m} & , \quad m > 0 \\ -(-m)^{-1/m} & , \quad m < 0 \\ 0 & , \quad m = 0 \end{cases}$$

in (2.3). For the third case we take

$$\mu = m \ (0, \pm 1, \pm 2, \ldots)$$

in (2.3)

<u>Case I</u>: We choose the control limits $C_* = -5.0$ and $C^* = 5.0$ with $\alpha = 1.0$, the following values of $Gs(.|.)$ are chosen.

$$G(7|C_*) = G(-7|C^*) = 0.4$$

$$G(6|C_*) = G(5|C_*) = G(-6|C^*) = G(-5|C^*) = 0.2$$

$$G(4|C_*) = G(3|C_*) = G(-4|C^*) = G(-3C^*) = 0.1$$

$$G(0|(C^*-C_*) = 1.0$$

$$G(.|.) = 0, \text{ otherwise}$$

Table 1 gives the transition probability matrix $[p_{ij}] = [\xi_{j-i}(i)]$ and Table 2 the values of δ_r, Δ_r, δ_r and Δ_r for $r = 0, \pm 1, \pm 2, \ldots$

It is seen that

$$\Sigma\, r\, \delta_{-r} \; > \; \Sigma\, r\, \Delta_r \quad \text{and} \quad \Sigma\, r\, \delta'_r \; > \; \Sigma\, r\, \Delta'_{-r}$$

Thus the condition (3.10) is satisfied and so the controller induces stability in the system. The steady state probability π_m of the process mean and the steady state distribution function F(x) of the observations are presented in table 3 and 4 respectively.

<u>Case II</u>: We choose the control limits $C_* = 04.0$, $C^* = 4.0$ and $\alpha = 0.5$. The values of Gx(.|.) are taken as follows:

$$G(3|C_*) = G(-3|C^*) = G(0|(C^*-C_*)) = 1.0$$

$$G(.|.) = 0.0, \text{ otherwise.}$$

The transition probability matrix and the values of δ_r, Δ_r, δ'_r and Δ'_r for $r = 0, \pm1, \pm2, \ldots$ are given in table 5 and table 6 respectively. Condition (3.10) is again satisfied, and the controller induces stability in the system. The steady state probabilities π_m of the process mean the steady state distribution of the observations are presented in table 7 and table 8 respectively.

Case III: We take the control limits $C_* = 3.0$, $C^* = 3.0$ $\alpha = 1.0$, and

$$G(9|C_*) = G(-9|C^*) = G(0|(C^*-C_*)) = 1.0,$$

$$G(.|.) = 0.0, \text{ otherwise.}$$

Table 9 gives the transition probability matrix. In this case condition (3.10) is violated and it is observed that the two-sided controller fails to induce stability into the process.

We have considered some other sets of values of G(.|.). In every situation we have found the condition (3.10) is violated and also that the 'two-sided controller' fails to induce stability. It appears that the condition (3.10) is not only sufficient but also necessary for the stability of the process.

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	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
-15	.233	.199	.088	.017	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-14	.191	.233	.198	.087	.017	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-13	.137	.192	.234	.197	.086	.016	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-12	.087	.137	.192	.234	.196	.085	.016	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-11	.041	.088	.138	.193	.234	.194	.083	.016	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-10	.011	.042	.088	.139	.194	.234	.193	.082	.015	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-9	.001	.011	.042	.089	.140	.195	.234	.191	.080	.015	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-8	.000	.002	.012	.043	.090	.141	.196	.234	.189	.078	.014	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
-7	.002	.007	.010	.016	.044	.089	.139	.193	.229	.183	.074	.013	.001	.000	.000	.000	.000	.000	.000	.000	.000
-6	.002	.017	.050	.059	.037	.043	.078	.121	.167	.198	.155	.061	.011	.001	.000	.000	.000	.000	.000	.000	.000
-5	.000	.008	.057	.160	.180	.086	.037	.048	.073	.100	.117	.091	.035	.006	.000	.000	.000	.000	.000	.000	.000
-4	.000	.001	.015	.101	.275	.298	.131	.030	.017	.023	.032	.037	.028	.011	.002	.000	.000	.000	.000	.000	.000
-3	.000	.000	.001	.018	.122	.324	.342	.144	.025	.004	.003	.005	.005	.004	.001	.000	.000	.000	.000	.000	.000
-2	.000	.000	.000	.001	.017	.120	.327	.353	.152	.026	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-1	.000	.000	.000	.000	.000	.006	.061	.242	.383	.242	.061	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000
0	.000	.000	.000	.000	.000	.000	.000	.006	.061	.242	.383	.242	.061	.006	.000	.000	.000	.000	.000	.000	.000
1	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006	.061	.242	.383	.242	.061	.006	.000	.000	.000	.000	.000
2	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.002	.026	.152	.353	.327	.120	.017	.001	.000	.000	.000
3	.000	.000	.000	.000	.000	.000	.001	.004	.005	.005	.003	.004	.025	.144	.342	.324	.122	.018	.001	.000	.000
4	.000	.000	.000	.000	.000	.000	.002	.011	.028	.037	.032	.023	.017	.030	.131	.298	.275	.101	.015	.001	.000
5	.000	.000	.000	.000	.000	.000	.000	.006	.035	.091	.117	.100	.073	.048	.037	.086	.180	.160	.057	.008	.000
6	.000	.000	.000	.000	.000	.000	.000	.001	.011	.061	.155	.198	.167	.121	.078	.043	.037	.059	.050	.017	.002
7	.000	.000	.000	.000	.000	.000	.000	.000	.001	.013	.074	.183	.229	.193	.139	.089	.044	.016	.010	.007	.002
8	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.014	.078	.189	.234	.196	.141	.090	.043	.012	.002	.000
9	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.015	.080	.191	.234	.195	.140	.089	.042	.011	.001
10	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.015	.082	.193	.234	.194	.139	.088	.042	.011
11	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.016	.083	.194	.234	.193	.138	.088	.041
12	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.016	.085	.196	.234	.192	.137	.087
13	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.016	.086	.197	.234	.192	.137
14	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.017	.087	.198	.233	.191
15	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.001	.017	.088	.199	.233

TABLE 1
TRANSITION PROBABILITY MATRIX

TABLE 2

VALUES OF SUPREMA AND INFIMA: Due to Symmetry $\delta_{-r} = \delta'_r$, $\Delta_r = \Delta'_{-r}$.

	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
δ_r	.001	.011	.061	.155	.198	.167	.121	.078	.037	.009	.001	.000	.000	.000	.000
Δ_r	.002	.019	.094	.203	.234	.196	.141	.090	.043	.037	.059	.050	.017	.002	.000

TABLE 3

STEADY STATE PROBABILITY OF THE PROCESS MEANS

$\pm m$	0	1	2	3	4	5	6	7
π_m	.07	.18	.12	.08	.05	.02	.01	.00

TABLE 4

STEADY STATE PROBABILITY OF THE OBSERVATIONS: $F(-x) = 1 - F(x)$, due to symmetry

x	0.0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
F(x)	.4985	.5630	.6286	.6942	.7563	.8117	.8586	.8967	.9271	.9501	.9669	.9785

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
-15	.196	.166	.109	.056	.023	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-14	.183	.196	.165	.109	.056	.023	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-13	.133	.182	.195	.164	.107	.055	.022	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-12	.077	.132	.180	.192	.160	.105	.054	.022	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-11	.040	.076	.128	.172	.183	.152	.099	.051	.020	.006	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-10	.036	.047	.075	.118	.156	.165	.137	.089	.045	.018	.006	.001	.000	.000	.000	-.00	.000	.000	.000	.000	.000
-9	.063	.059	.057	.071	.101	.130	.136	.112	.072	.037	.015	.005	.001	.000	.000	.000	.000	-.00	.000	.000	.000
-8	.093	.099	.087	.070	.066	.079	.096	.098	.080	.052	.026	.010	.003	.001	.000	.000	.000	.000	.000	.000	.000
-7	.097	.129	.136	.114	.082	.061	.057	.062	.061	.049	.032	.016	.006	.002	.000	.000	.000	.000	.000	.000	.000
-6	.070	.119	.158	.164	.135	.091	.055	.039	.035	.032	.025	.016	.008	.003	.001	.000	.000	.000	.000	.000	.000
-5	.037	.079	.134	.176	.181	.147	.095	.051	.027	.018	.014	.011	.007	.003	.001	.000	.000	.000	.000	.000	-.00
-4	.015	.040	.085	.142	.185	.189	.151	.095	.048	.021	.010	.006	.004	.002	.001	.000	.000	.000	.000	.000	.000
-3	.004	.015	.042	.088	.146	.189	.192	.152	.095	.047	.018	.006	.002	.001	.001	.000	.000	.000	.000	.000	.000
-2	.001	.004	.015	.041	.087	.145	.189	.193	.154	.097	.047	.018	.006	.002	.000	.000	.000	.000	.000	.000	.000
-1	.000	.000	.002	.009	.028	.066	.121	.175	.197	.175	.121	.066	.028	.009	.002	.001	.000	.000	.000	.000	.000
0	.000	.000	.000	.000	.002	.009	.028	.066	.121	.175	.197	.175	.121	.066	.028	.009	.002	.000	.000	.000	.000
1	.000	.000	.000	.000	.000	.001	.002	.009	.028	.066	.121	.175	.197	.175	.121	.066	.028	.009	.002	.000	.000
2	.000	.000	.000	.000	.000	.000	.000	.002	.006	.018	.047	.097	.154	.193	.189	.145	.087	.041	.015	.004	.001
3	.000	.000	.000	.000	.000	.000	.001	.001	.002	.006	.018	.047	.095	.152	.192	.189	.146	.088	.042	.015	.004
4	.000	.000	.000	.000	.000	.000	.001	.002	.004	.006	.010	.021	.048	.095	.151	.189	.185	.142	.085	.040	.015
5	-.00	.000	.000	.000	.000	.000	.001	.003	.007	.011	.014	.018	.027	.051	.095	.147	.181	.176	.134	.079	.037
6	.000	.000	.000	.000	.000	.000	.001	.003	.008	.016	.025	.032	.035	.039	.055	.091	.135	.164	.158	.119	.070
7	.000	.000	.000	.000	.000	.000	.000	.002	.006	.016	.032	.049	.061	.062	.057	.061	.082	.114	.136	.129	.097
8	.000	.000	.000	.000	.000	.000	.000	.001	.003	.010	.026	.052	.080	.098	.096	.079	.066	.070	.087	.099	.093
9	.000	.000	.000	-.00	.000	.000	.000	.000	.001	.005	.015	.037	.072	.112	.136	.130	.101	.071	.057	.059	.063
10	.000	.000	.000	.000	.000	-.00	.000	.000	.000	.001	.006	.018	.045	.089	.137	.165	.156	.118	.075	.047	.036
11	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.002	.006	.020	.051	.099	.152	.183	.172	.128	.076	.040
12	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.002	.007	.022	.054	.105	.160	.192	.180	.132	.077
13	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.002	.007	.022	.055	.107	.164	.195	.182	.133
14	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.002	.007	.023	.056	.109	.165	.196	.183
15	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.002	.007	.023	.056	.109	.166	.196

TABLE 5
TRANSITION PROBABILITY MATRIX

TABLE 6

VALUES OF SUPREMA AND INFIMA : Due to symmetry $\delta_{-r} = \delta'_r$, $\Delta_r = \Delta'_{-r}$

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
δ_r	.005	.015	.037	.072	.112	.136	.130	.101	.071	.032	.011	.003	.001	.000	.000	.000	.000	.000	.000
Δ_r	.008	.024	.059	.113	.168	.197	.183	.133	.077	.057	.059	.063	.058	.042	.025	.011	.004	.001	.000

TABLE 7

STEADY STATE PROBABILITY OF THE PROCESS MEANS

$\pm$ m	0	1	2	3	4	5	6
π_m	.01	.22	.15	.08	.03	.01	.00

TABLE 8

STEADY STATE PROBABILITY OF THE OBSERVATIONS: $F(-x) = 1 - F(x)$, Due to symmetry

x	0.0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
F(x)	.5000	.5630	.6248	.6841	.7398	.7907	.8359	.8747	.9070	.9329	.9530	.9681

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
-15	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-14	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-13	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-12	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-11	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-10	.242	.061	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-9	.242	.383	.242	.061	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-8	.006	.061	.242	.383	.242	.061	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-7	.000	.000	.006	.061	.242	.383	.242	.061	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-6	.000	.000	.000	.000	.006	.061	.241	.382	.241	.061	.006	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
-5	.009	.005	.001	.000	.000	.000	.006	.059	.236	.374	.236	.059	.006	.000	.000	.000	.000	.000	.000	.000	.000
-4	.010	.038	.061	.038	.010	.001	.000	.000	.005	.051	.203	.322	.203	.051	.005	.000	.000	.000	.000	.000	.000
-3	.000	.003	.030	.121	.191	.121	.030	.003	.000	.000	.003	.030	.121	.191	.121	.030	.003	.000	.000	.000	.000
-2	.000	.000	.000	.005	.051	.203	.322	.203	.051	.005	.000	.000	.001	.010	.038	.061	.038	.010	.001	.000	.000
-1	.000	.000	.000	.000	.000	.006	.059	.236	.374	.236	.059	.006	.000	.000	.000	.001	.005	.009	.005	.001	.000
0	.000	.001	.000	.000	.000	.000	.000	.006	.060	.241	.382	.241	.060	.006	.000	.000	.000	.000	.000	.001	.000
1	.000	.001	.005	.009	.005	.001	.000	.000	.000	.006	.059	.236	.374	.236	.059	.006	.000	.000	.000	.000	.000
2	.000	.000	.001	.010	.038	.061	.038	.010	.001	.000	.000	.005	.051	.203	.322	.203	.051	.005	.000	.000	.000
3	.000	.000	.000	.000	.003	.030	.121	.191	.121	.030	.003	.000	.000	.003	.030	.121	.191	.121	.030	.003	.000
4	.000	.000	.000	.000	.000	.000	.005	.051	.203	.322	.203	.051	.005	.000	.000	.001	.010	.038	.061	.038	.010
5	.000	.000	.000	.000	.000	.000	.000	.000	.006	.059	.236	.374	.236	.059	.006	.000	.000	.000	.001	.005	.009
6	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006	.061	.241	.382	.241	.061	.006	.000	.000	.000	.000
7	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006	.061	.242	.383	.242	.061	.006	.000	.000
8	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006	.061	.242	.383	.242	.061	.006
9	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006	.061	.242	.383	.242
10	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006	.061	.242
11	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.006
12	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
13	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
14	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
15	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

TABLE 9
TRANSITION PROBABILITY MATRIX

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